Lecture 7

Calculation of curvilinear integrals

We restrict to the case of 2D plane $\mathbb{R}^2$.

Let we have a curve L:

$$L = \{ (x, y) \mid x = x(s),\ y = y(s) \},$$

where s is the arc-length of the curve.

Let $F(x,y)$ be a function defined on L (e.g. a density of the mass)

<u>Definition</u>. The curvilinear integral is defined by the equality

$$\int_L F(x,y)\,ds \stackrel{def}{=} \int_0^S F(x(s), y(s))\,ds.$$

It is called a line integral of the first kind.

The definition is described in terms of Riemann sums. This approach gives conditions to define a set of $F(x,y)$ and L which are Riemann-integrable.

1. Define a partition τ of L for $s \in [0, S]$, $\{s_i : i = 0, 1, \dots, m\}$

$\Delta s_i = s_i - s_{i-1}$ is the length of the section of L from the point $(x(s_{i-1}), y(s_{i-1}))$ to the point $(x(s_i), y(s_i))$.

$$\delta_\tau = \max_{1 < i \le m} \Delta s_i$$

2. Next we select $\xi_i \in [s_{i-1}, s_i]$ a sample point

3. A Riemann sum is defined by

$$\sigma_\tau \stackrel{def}{=} \sum_{i=1}^{m} F(x(\xi_i), y(\xi_i)) \Delta s_i .$$

<u>Def</u>.

$$\int_L F(x,y)\, ds = \lim_{\delta_\tau \to 0} \sigma_\tau$$

if this limit do not depend on mesh τ and sample points ξ_i.

If curve L is defined in 3D space and $F(x, y, z)$ is a smooth function

then

$$\int_L f(x, y, z)\, ds = \lim_{\tau \to 0} \sum_{i=1}^{n} F(x_i, y_i, z_i) \Delta s_i$$

<u>Calculation of integrals.</u>

We calculated in previous lectures how the length of curve line is expressed.

Let us assume that L is defined ~~as~~ in plane xOy by

$$y = y(x) \quad , \quad a \le x \le b.$$

Then

$$\Delta s_i \approx \sqrt{1 + (y'(x))^2}\, \Delta x_i$$

$$\sigma_\tau = \sum_{i=1}^{n} F(x_i, y_i)\sqrt{1+(y'(x_i))^2}\,\Delta x_i$$

Taking a limit $\delta_\tau = \max \Delta x_i \to 0$

we get

$$\int_L F(x,y)\,ds = \int_a^b F(x, y(x))\sqrt{1+(y'(x))^2}\,dx$$

If L is defined by parametric equations

$$x = x(t), \quad y = y(t), \quad t \in [c, d]$$

then

$$\int_L F(x,y)\,ds = \int_c^d F(x(t), y(t))\sqrt{(x'_t)^2+(y'_t)^2}\,dt$$

$$\boxed{ds = \sqrt{(x'_t)^2+(y'_t)^2}\,dt}$$

If L is defined in 3D space

$$x = x(t), \quad y = y(t), \quad z = z(t), \quad t \in [c, d]$$

then the curvilinear integral is calculated by the formula

$$\int_L F(x,y,z)\,ds = \int_c^d F(x(t),y(t),z(t))\sqrt{x_t'^2+y_t'^2+z_t'^2}\,dt$$

Remark. This integral of first kind does not depend on the orientation of the curve L.

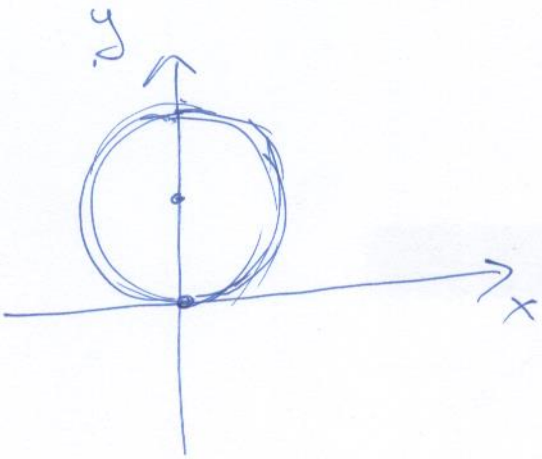
$$\int_{AB} F(x,y)\,ds = \int_{BA} F(x,y)\,ds.$$

Example 1. Calculate

$$\int_L (x+y)\,ds,$$

when L is defined by the formula

$L: x^2+y^2 = 2y$



We use polar coordinates:

$x = r\cos\varphi$

$y = r\sin\varphi$

Curve L is defined in polar coordinates by

$$L: \quad r^2\cos^2\varphi + r^2\sin^2\varphi = 2r\sin\varphi$$

$$r^2 = 2r\sin\varphi \Rightarrow \boxed{\begin{array}{l} r = 2\sin\varphi \\ 0 \le \varphi \le \pi \end{array}}$$

$$ds = \sqrt{(x'_\varphi)^2 + (y'_\varphi)^2}\, d\varphi$$

$$= \sqrt{(r'_\varphi\cos\varphi - r\sin\varphi)^2 + (r'_\varphi\sin\varphi + r\cos\varphi)^2}\, d\varphi$$

$$= \sqrt{r^2 + (r'_\varphi)^2}\, d\varphi = 2\, d\varphi.$$

$$\int_L (x+y)\,ds = 2\int_0^{\pi} \left(r\cos\varphi + r\sin\varphi\right) d\varphi$$

$$= 4\int_0^{\pi} \left(\sin^2\varphi + \sin\varphi\cos\varphi\right) d\varphi = 2\pi.$$

Curvilinear integrals of second kind

Let us consider the curve L defined in $\mathcal{D} \subset \mathbb{R}^2$.

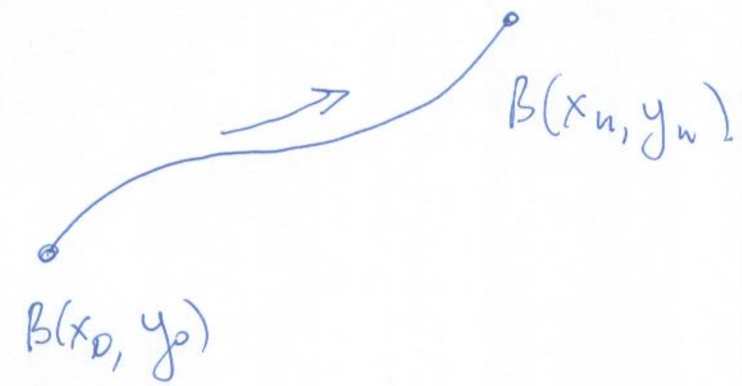
Also we have a vector $\vec{F}$ (a field of forces)

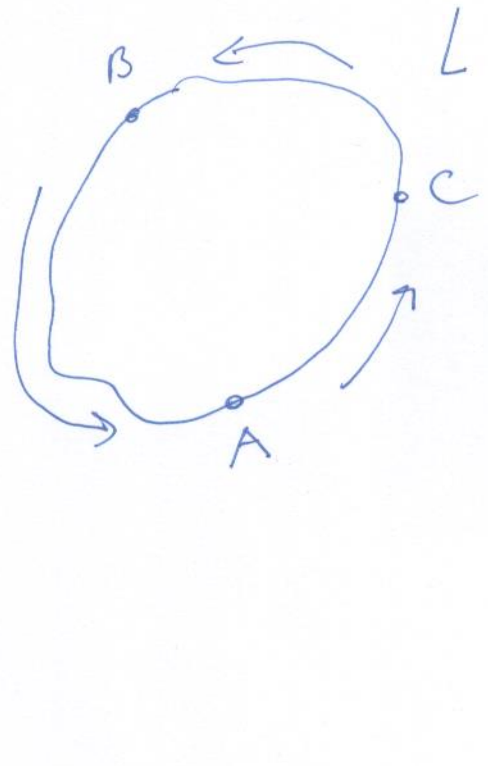
$$\vec{F} = P(x,y)\,\vec{i} + Q(x,y)\,\vec{j}$$

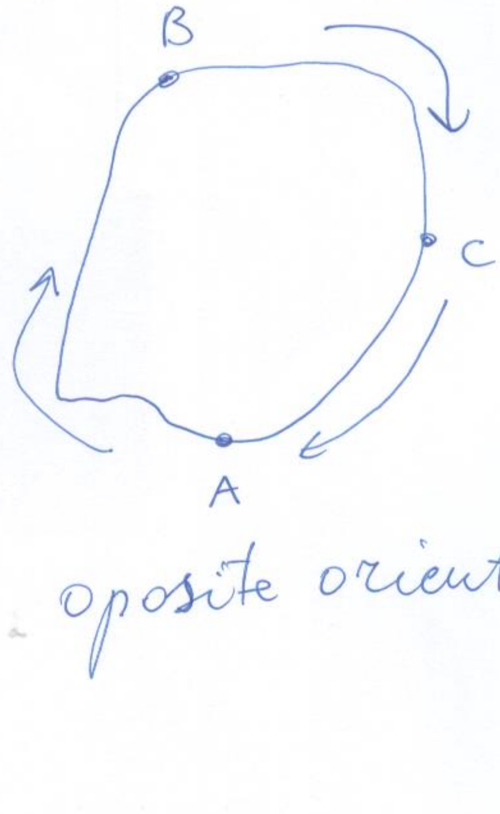
A material point $M(x,y)$ is ~~moo~~ moved due to influence of forces $\vec{F}$ from a starting point of L $B(x_0, y_0)$ to the final point $B(x_n, y_n) \in L$.

We want ~~to~~ calculate the work W done by force $\vec{F}$.

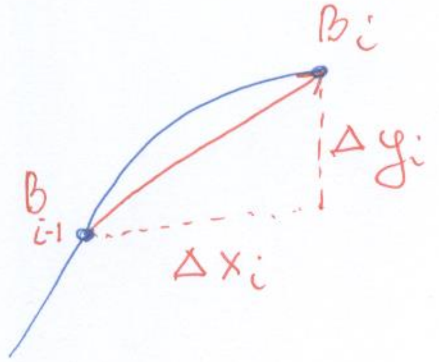
1. The orientation of the curve is given (e.g $B(x_0, y_0) \to B(x_n, y_n)$).







oposite orientation.



$$\vec{\Delta S_i} = (\Delta x_i, \Delta y_i).$$

Then $W_i \approx \vec{F}(\bar{x}_i, \bar{y}_i) \cdot \Delta S_i$

$$= P(\bar{x}_i, \bar{y}_i)\Delta x_i + Q(\bar{x}_i, \bar{y}_i)\Delta y_i$$

Let us consider some partition of L and calculate a sum

$$\sigma_\tau = \sum_{i=1}^{n} P(\bar{x}_i, \bar{y}_i)\Delta x_i + Q(\bar{x}_i, \bar{y}_i)\Delta y_i$$

<u>Def</u>. Curvilinear integral of second kind is defined by a limit:

$$\int_L P(x,y)\,dx + Q(x,y)\,dy = \lim_{\delta_\tau \to 0} \sigma_\tau .$$

If L is defined in 3D space and vector of forces is given by

$$\vec{F} = P(x,y,z)\,\vec{i} + Q(x,y,z)\,\vec{j} + R(x,y,z)\,\vec{k}$$

then

$$W = \int_L P(x,y,z)\,dx + Q(x,y,z)\,dy + R(x,y,z)\,dz$$

<u>Definition</u>. If L is a closed curve, then we use notation (here L is defined in a plane)

$$\oint_L P(x,y)\,dx + Q(x,y)\,dy$$

or in space $\mathbb{R}^3$:

$$\oint_L P(x,y,z)\,dx + Q(x,y,z)\,dy + R(x,y,z)\,dz.$$

If the orientation of L is changed then

$$\int_{AB} P(x,y)\,dx + Q(x,y)\,dy = -\int_{BA} P(x,y)\,dx + Q(x,y)\,dy.$$

Calculation of integrals

Our aim to reduce curvilinear integrals of second kind to definite integrals.

Let us consider integral

$$\int_{AB} P(x,y)\,dx$$

A curve L is defined by parametric equations

$$x = x(t), \quad y = y(t), \quad t_0 \le t \le T$$

$$\boxed{\int_{AB} P(x,y)\,dx = \int_{t_0}^{T} P(x(t), y(t))\, x'(t)\,dt}$$

$$\sigma_\tau = \sum_{i=1}^{n} P(\bar{x}_i, \bar{y}_i)\,\Delta x_i$$

$$\Delta x_i = x(t_i) - x(t_{i-1}) = x'(\tilde{t}_i)\,\Delta t_i$$

We can select $(\bar{x}_i, \bar{y}_i)$ such that

$$\bar{x}_i = x(\tilde{t}_i), \quad \bar{y}_i = y(\tilde{t}_i).$$

Then

$$\sigma_\tau = \sum_{i=1}^{n} P(x(\tilde{t}_i), y(\tilde{t}_i))\, x'(\tilde{t}_i) \Delta t_i.$$

By taking a limit $\max \Delta t_i \to 0$ we get the required formula

$$\int_{AB} P(x,y)\,dx = \int_{t_0}^{T} P(x(t), y(t))\, x'(t)\, dt.$$

Similarly it follows that

$$\int_{AB} P(x,y)\,dx + Q(x,y)\,dy = \int_{t_0}^{T} \Big(P(x(t), y(t))\, x'(t)$$
$$+ Q(x(t), y(t))\, y'(t) \Big)\, dt.$$

Thus integrals exist if P, Q are continuous and L is smooth.

Example 1. Calculate

$$W_1 = \int_L xy\,dx + (x^2 - y^3)\,dy$$

from point $O(0,0)$ till $A(1,1)$,

$L: \; y = x^3$.

Parametric equation $y = y(x)$.

$$dy = 3x^2\,dx$$

$$W = \int_0^1 x\,x^3\,dx + \int_0^1 (x^2 - x^9)\,3x^2\,dx = \frac{11}{20}.$$

b) $y = x$ defines L

$$dy = dx$$

$$W_2 = \int_0^1 x^2\,dx + \int_0^1 (x^2 - x^3)\,dx = \frac{5}{12}.$$

$$W_1 \neq W_2.$$

Example 2. Calculate

$$W = \int_L (2x + 3y^2)\,dx + (6xy - 1)\,dy$$

from point $A(1,1)$ till poin $B(2,4)$ when curves L are defined by:

a) a line $y = 3x - 2$

b) a parabola $y = x^2$

c) a curve ACB, where $C(1,2)$.

Solution.

a) $y = 3x - 2$ $\quad dy = 3dx$

$$W = \int_1^2 \left(2x + 3(3x-2)^2\right)dx + \int_1^2 \left(6x(3x-2) - 1\right) \times 3dx = 93$$

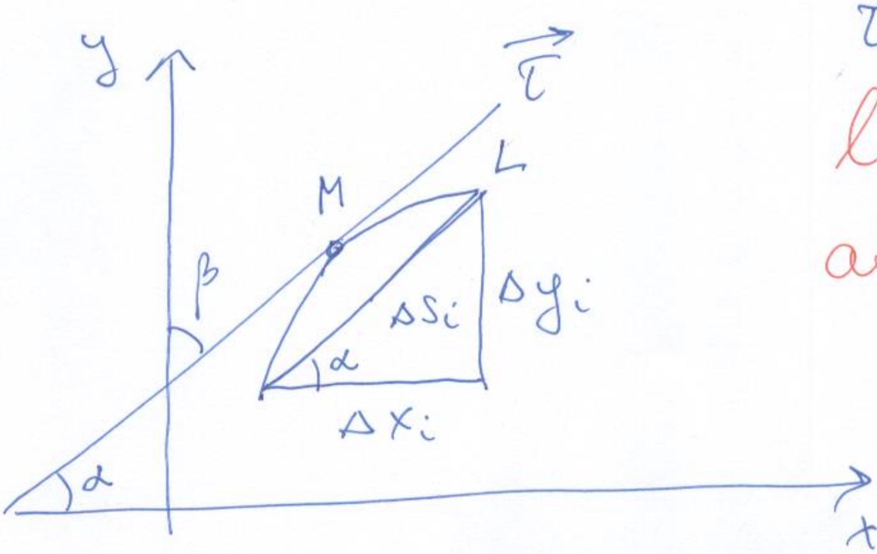
b) ~~W~~= $dy = 2x\,dx$

$$W = \int_1^2 (2x + 3x^4)\,dx + \int_1^2 (6x^3 - 1)\,2x\,dx = 93$$

c) $W = 93$!!!

The value of the integral is the same for any path L connecting points A and B.

Connection of integrals of both kinds.



$\vec{\tau}$ is tangent line of a curve L at a point M.

We get approximate equalities

$$\Delta x_i \approx \cos\alpha \, \Delta S_i$$

$$\Delta y_i \approx \cos\beta \, \Delta S_i$$

Take sums :

$$\sum_{i=1}^{n} P(\bar{x}_i, \bar{y}_i)\Delta x_i + Q(\bar{x}_i, \bar{y}_i)\Delta y_i$$

$$\approx \sum_{i=1}^{n} \left[\left(P(\bar{x}_i, \bar{y}_i)\cos\alpha + Q(\bar{x}_i, \bar{y}_i)\cos\beta \right] \Delta s_i$$

By takin a limit $\Delta s_i \to 0$ we get

$$\int_L P(x,y)\,dx + Q(x,y)\,dy = \int_L \left(P(x,y)\cos\alpha + Q\cos\beta \right) ds$$

$$\alpha = \alpha(x,y), \quad \beta = \beta(x,y)$$

In the case of 3D integrals

$$\int_L P(x,y,z)\,dx + Q(x,y,z)\,dy + R(x,y,z)\,ds$$

$$= \int_L \left(P(x,y,z)\cos\alpha + Q(x,y,z)\cos\beta + R(x,y,Z)\cos\gamma \right) ds$$

Green's theorem

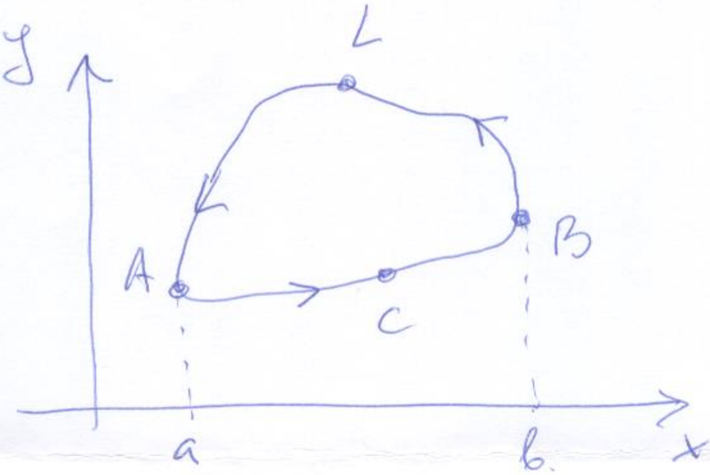
This theorem relates a line integral around a simple closed curve L to a double integral over the plane region D bounded by L.

Th. Let L be positively oriented, smooth and simple closed curve in a plane and D be the region bounded by L. If $P(x,y)$ and $Q(x,y)$ are functions of (x,y) defined on D and have continuous partial derivatives, then

$$\oint_L (P\,dx + Q\,dy) = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$$

where the path of integration along L is anticlockwise.

The curve is said to be positively oriented (or counterclockwise oriented) if one always has the curve interior to the left.



L is defined by $y = y_1(x)$ for ACB

$y = y_2(x)$ for BLA.

A double integral is changed to a sequence of 1D integrals (iterated)

$$\iint_D \frac{\partial P}{\partial y}\,dx\,dy = \int_a^b dx \int_{y_1(x)}^{y_2(x)} \frac{\partial P}{\partial y}\,dy$$

$$= \int_a^b \Big(P(x, y_2(x)) - P(x, y_1(x)) \Big)\,dx$$

$$= \int_a^b P(x, y_2(x))\,dx - \int_a^b P(x, y_1(x))\,dx$$

(parametric formula)

$$= \int_{AEB} P(x,y)\,dx - \int_{ACB} P(x,y)\,dx$$

$$= -\left(\int_{BEA} P(x,y)\,dx + \int_{ACB} P(x,y)\,dx \right)$$

$$= -\oint_L P(x,y)\,dx$$

Similarly we get

$$\iint_D \frac{\partial Q}{\partial x}\,dx\,dy = \oint_L Q(x,y)\,dy$$

Example. Calculate

$$W = \oint_L xy\,dx + (x^2+y^2)\,dy,$$

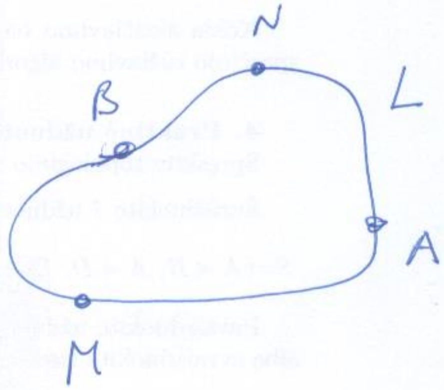
when L is a circle

$$x^2 + y^2 = 2x$$

and orientation of L is positive direction

a) calculate directly ($W = \pi$)

b) Use Green's formula and calculate a double integral.



Conclusion

If

$$\oint_L P\,dx + Q\,dy = 0$$

then we get

$$\int_{MAN} P\,dx + Q\,dy + \underbrace{\int_{NBM} P\,dx + Q\,dy}_{= -\int_{MBN} P\,dx + Q\,dy} = 0$$

$$\int_{MAN} P\,dx + Q\,dy = \int_{MBN} P\,dx + Q\,dy \quad !$$

It follows from Green's theorem that

$$\oint_L P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx\,dy.$$

For any L (and D)

$$\boxed{\oint_L P\,dx + Q\,dy = 0 \iff \frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}}$$

In our previous example we had:

$$P(x,y) = 2x + 3y^2$$

$$Q(x,y) = 6xy - 1$$

$$\frac{\partial P}{\partial y} = 6y \quad \text{and} \quad \frac{\partial Q}{\partial x} = 6y$$